Singular Integrals of Subordinators and Applications to SPDEs

邓昌松

武汉大学

@中南大学
2021年7月

Content

Base on a joint work with R. Schilling (Dresden) and L. Xu (Macau): arXiv: 2009.04785

1. **Background and motivation**
2. **A zero-one law**
3. **Moment estimates**
4. **Applications to SPDEs**

$$\int_{0+} t^{-\theta}\,\mathrm{d}t \quad \begin{cases} < \infty, & \text{if } \theta < 1, \\ = \infty, & \text{if } \theta \geq 1. \end{cases}$$

If S_t is an α-stable subordinator $(0 < \alpha < 1)$,

$$\int_{0+} t^{-\theta}\,\mathrm{d}S_t \quad \begin{cases} < \infty, & \text{if } \theta < 1/\alpha, \\ = \infty, & \text{if } \theta \geq 1/\alpha. \end{cases}$$

See L. Xu (2018).

Background

$$\int_{0+} t^{-\theta}\,\mathrm{d}t \quad \begin{cases} < \infty, & \text{if } \theta < 1, \\ = \infty, & \text{if } \theta \geq 1. \end{cases}$$

If S_t is an α-stable subordinator $(0 < \alpha < 1)$,

$$\int_{0+} t^{-\theta}\,\mathrm{d}S_t \quad \begin{cases} < \infty, & \text{if } \theta < 1/\alpha, \\ = \infty, & \text{if } \theta \geq 1/\alpha. \end{cases}$$

See L. Xu (2018).

$$\int_{0+} t^{-\theta}\,\mathrm{d}t \quad \begin{cases} < \infty, & \text{if } \theta < 1, \\ = \infty, & \text{if } \theta \geq 1. \end{cases}$$

If S_t is an α-stable subordinator $(0 < \alpha < 1)$,

$$\int_{0+} t^{-\theta}\,\mathrm{d}S_t \quad \begin{cases} < \infty, & \text{if } \theta < 1/\alpha, \\ = \infty, & \text{if } \theta \geq 1/\alpha. \end{cases}$$

See L. Xu (2018).

Background

$$\int_{0+} t^{-\theta}\, \mathrm{d}t \quad \begin{cases} < \infty, & \text{if } \theta < 1, \\ = \infty, & \text{if } \theta \geq 1. \end{cases}$$

If S_t is an α-stable subordinator $(0 < \alpha < 1)$,

$$\int_{0+} t^{-\theta}\, \mathrm{d}S_t \quad \begin{cases} < \infty, & \text{if } \theta < 1/\alpha, \\ = \infty, & \text{if } \theta \geq 1/\alpha. \end{cases}$$

See L. Xu (2018).

Background

$$\int_{0+} t^{-\theta}\,\mathrm{d}t \quad \begin{cases} < \infty, & \text{if } \theta < 1, \\ = \infty, & \text{if } \theta \geq 1. \end{cases}$$

If S_t is an α-stable subordinator $(0 < \alpha < 1)$,

$$\int_{0+} t^{-\theta}\,\mathrm{d}S_t \quad \begin{cases} < \infty, & \text{if } \theta < 1/\alpha, \\ = \infty, & \text{if } \theta \geq 1/\alpha. \end{cases}$$

See L. Xu (2018).

Motivation

- Consider the SPDE

$$\mathrm{d}X_t = -AX_t\mathrm{d}t + F(X_t)\mathrm{d}t + Q(X_{t-})\mathrm{d}W_{S_t}.$$

- Solution

$$X_t = \mathrm{e}^{-At}X_0 + \int_0^t \mathrm{e}^{-(t-s)A}F(X_s)\,\mathrm{d}s + \underbrace{\int_0^t \mathrm{e}^{-(t-s)A}Q(X_{s-})\,\mathrm{d}W_{S_s}}_{=:Z_t}$$

$$\mathbb{E}|A^\theta Z_T|^p \le \cdots \le C\mathbb{E}\left(\int_0^T t^{-2\theta}\,\mathrm{d}S_t\right)^p.$$

- Consider the SPDE

$$\mathrm{d}X_t = -AX_t\mathrm{d}t + F(X_t)\mathrm{d}t + Q(X_{t-})\mathrm{d}W_{S_t}.$$

- Solution

$$X_t = \mathrm{e}^{-At}X_0 + \int_0^t \mathrm{e}^{-(t-s)A}F(X_s)\,\mathrm{d}s + \underbrace{\int_0^t \mathrm{e}^{-(t-s)A}Q(X_{s-})\,\mathrm{d}W_{S_s}}_{=:Z_t}$$

$$\mathbb{E}|A^\theta Z_T|^p \le \cdots \le C\mathbb{E}\left(\int_0^T t^{-2\theta}\,\mathrm{d}S_t\right)^p.$$

Motivation

- Consider the SPDE

$$\mathrm{d}X_t = -AX_t\mathrm{d}t + F(X_t)\mathrm{d}t + Q(X_{t-})\mathrm{d}W_{S_t}.$$

- Solution

$$X_t = \mathrm{e}^{-At}X_0 + \int_0^t \mathrm{e}^{-(t-s)A}F(X_s)\,\mathrm{d}s + \underbrace{\int_0^t \mathrm{e}^{-(t-s)A}Q(X_{s-})\,\mathrm{d}W_{S_s}}_{=:Z_t}$$

$$\mathbb{E}|A^\theta Z_T|^p \le \cdots \le C\mathbb{E}\left(\int_0^T t^{-2\theta}\,\mathrm{d}S_t\right)^p.$$

Motivation

- Consider the SPDE

$$\mathrm{d}X_t = -AX_t\mathrm{d}t + F(X_t)\mathrm{d}t + Q(X_{t-})\mathrm{d}W_{S_t}.$$

- Solution

$$X_t = \mathrm{e}^{-At}X_0 + \int_0^t \mathrm{e}^{-(t-s)A}F(X_s)\,\mathrm{d}s + \underbrace{\int_0^t \mathrm{e}^{-(t-s)A}Q(X_{s-})\,\mathrm{d}W_{S_s}}_{=:Z_t}$$

$$\mathbb{E}|A^\theta Z_T|^p \le \cdots \le C\mathbb{E}\left(\int_0^T t^{-2\theta}\,\mathrm{d}S_t\right)^p.$$

Motivation

- Consider the SPDE

$$\mathrm{d}X_t = -AX_t\mathrm{d}t + F(X_t)\mathrm{d}t + Q(X_{t-})\mathrm{d}W_{S_t}.$$

- Solution

$$X_t = \mathrm{e}^{-At}X_0 + \int_0^t \mathrm{e}^{-(t-s)A}F(X_s)\,\mathrm{d}s + \underbrace{\int_0^t \mathrm{e}^{-(t-s)A}Q(X_{s-})\,\mathrm{d}W_{S_s}}_{=:Z_t}$$

$$\mathbb{E}|A^\theta Z_T|^p \le \cdots \le C\mathbb{E}\left(\int_0^T t^{-2\theta}\,\mathrm{d}S_t\right)^p.$$

- Consider the SPDE

$$\mathrm{d}X_t = -AX_t\mathrm{d}t + F(X_t)\mathrm{d}t + Q(X_{t-})\mathrm{d}W_{S_t}.$$

- Solution

$$X_t = \mathrm{e}^{-At}X_0 + \int_0^t \mathrm{e}^{-(t-s)A}F(X_s)\,\mathrm{d}s + \underbrace{\int_0^t \mathrm{e}^{-(t-s)A}Q(X_{s-})\,\mathrm{d}W_{S_s}}_{=:Z_t}$$

$$\mathbb{E}|A^\theta Z_T|^p \le \cdots \le C\mathbb{E}\left(\int_0^T t^{-2\theta}\,\mathrm{d}S_t\right)^p.$$

Motivation

- Consider the SPDE

$$\mathrm{d}X_t = -AX_t\mathrm{d}t + F(X_t)\mathrm{d}t + Q(X_{t-})\mathrm{d}W_{S_t}.$$

- Solution

$$X_t = \mathrm{e}^{-At}X_0 + \int_0^t \mathrm{e}^{-(t-s)A}F(X_s)\,\mathrm{d}s + \underbrace{\int_0^t \mathrm{e}^{-(t-s)A}Q(X_{s-})\,\mathrm{d}W_{S_s}}_{=:Z_t}$$

$$\mathbb{E}|A^\theta Z_T|^p \le \cdots \le C\mathbb{E}\left(\int_0^T t^{-2\theta}\,\mathrm{d}S_t\right)^p.$$

Our question

Recall for α-stable subordinator, L. Xu (2018):

$$\int_0^T t^{-\theta}\,\mathrm{d}S_t \begin{cases} < \infty, & \text{if } \theta < 1/\alpha, \\ = \infty, & \text{if } \theta \geq 1/\alpha. \end{cases}$$

Question:

Q1: General subordinator?

Q2: General singularity?

e.g. $\int_0^T (T-t)^{-\theta}\,\mathrm{d}S_t, \quad \int_T^\infty t^{-\theta}\,\mathrm{d}S_t, \quad \cdots$

Our question

Recall for α-stable subordinator, L. Xu (2018):

$$\int_0^T t^{-\theta}\, \mathrm{d}S_t \begin{cases} < \infty, & \text{if } \theta < 1/\alpha, \\ = \infty, & \text{if } \theta \geq 1/\alpha. \end{cases}$$

Question:

Q1: General subordinator?

Q2: General singularity?

e.g. $\int_0^T (T-t)^{-\theta}\, \mathrm{d}S_t, \quad \int_T^\infty t^{-\theta}\, \mathrm{d}S_t, \quad \cdots$

Our question

Recall for α-stable subordinator, L. Xu (2018):

$$\int_0^T t^{-\theta}\,\mathrm{d}S_t \begin{cases} < \infty, & \text{if } \theta < 1/\alpha, \\ = \infty, & \text{if } \theta \geq 1/\alpha. \end{cases}$$

Question:

Q1: General subordinator?

Q2: General singularity?

e.g. $\int_0^T (T-t)^{-\theta}\,\mathrm{d}S_t, \quad \int_T^\infty t^{-\theta}\,\mathrm{d}S_t, \quad \cdots$

Our question

Recall for α-stable subordinator, L. Xu (2018):

$$\int_0^T t^{-\theta}\,\mathrm{d}S_t \begin{cases} < \infty, & \text{if } \theta < 1/\alpha, \\ = \infty, & \text{if } \theta \geq 1/\alpha. \end{cases}$$

Question:

Q1: General subordinator?

Q2: General singularity?

e.g. $\int_0^T (T-t)^{-\theta}\,\mathrm{d}S_t, \quad \int_T^\infty t^{-\theta}\,\mathrm{d}S_t, \quad \cdots$

Our question

Recall for α-stable subordinator, L. Xu (2018):

$$\int_0^T t^{-\theta}\,\mathrm{d}S_t \begin{cases} < \infty, & \text{if } \theta < 1/\alpha, \\ = \infty, & \text{if } \theta \geq 1/\alpha. \end{cases}$$

Question:

Q1: General subordinator?

Q2: General singularity?

e.g. $\int_0^T (T-t)^{-\theta}\,\mathrm{d}S_t, \quad \int_T^\infty t^{-\theta}\,\mathrm{d}S_t, \quad \cdots$

Our question (cont.)

Recall for α-stable subordinator, L. Xu (2018):

$$\mathbb{E}\left(\int_0^T t^{-\theta}\,\mathrm{d}S_t\right)^p \le C_{\alpha,\theta,p}\,\boldsymbol{T}^{p(1/\alpha-\theta)}, \quad \theta < 1/\alpha,\ p \in (0,\alpha),$$

$$\mathbb{E}\left(\int_0^T \mathrm{e}^{-\lambda t}\,\mathrm{d}S_t\right)^p \le C_{\alpha,\lambda,p}, \quad \lambda > 0,\ p \in (0,\alpha).$$

Question:

Q3: **Optimality** for the moment estimates?

$$\mathbb{E}\left(\int_0^T t^{-\theta}\,\mathrm{d}S_t\right)^p = \frac{\Gamma\left(1-\frac{p}{\alpha}\right)}{(1-\alpha\theta)^{p/\alpha}\Gamma(1-p)}\boldsymbol{T}^{p(1/\alpha-\theta)}, \quad \theta < 1/\alpha,\ p < \alpha,$$

$$\mathbb{E}\left(\int_0^T \mathrm{e}^{-\lambda t}\,\mathrm{d}S_t\right)^p = \frac{\Gamma\left(1-\frac{p}{\alpha}\right)}{\Gamma(1-p)}\left(\frac{1-\mathrm{e}^{-\alpha\lambda T}}{\alpha\lambda}\right)^{p/\alpha}, \quad \lambda > 0,\ p < \alpha.$$

Our question (cont.)

Recall for α-stable subordinator, L. Xu (2018):

$$\mathbb{E}\left(\int_0^T t^{-\theta}\,\mathrm{d}S_t\right)^p \le C_{\alpha,\theta,p}\,\boldsymbol{T}^{p(1/\alpha-\theta)}, \quad \theta < 1/\alpha,\ p \in (0,\alpha),$$

$$\mathbb{E}\left(\int_0^T \mathrm{e}^{-\lambda t}\,\mathrm{d}S_t\right)^p \le C_{\alpha,\lambda,p}, \quad \lambda > 0,\ p \in (0,\alpha).$$

Question:

Q3: **Optimality** for the moment estimates?

$$\mathbb{E}\left(\int_0^T t^{-\theta}\,\mathrm{d}S_t\right)^p = \frac{\Gamma\left(1-\frac{p}{\alpha}\right)}{(1-\alpha\theta)^{p/\alpha}\Gamma(1-p)}\,\boldsymbol{T}^{p(1/\alpha-\theta)}, \quad \theta < 1/\alpha,\ p < \alpha,$$

$$\mathbb{E}\left(\int_0^T \mathrm{e}^{-\lambda t}\,\mathrm{d}S_t\right)^p = \frac{\Gamma\left(1-\frac{p}{\alpha}\right)}{\Gamma(1-p)}\left(\frac{1-\mathrm{e}^{-\alpha\lambda T}}{\alpha\lambda}\right)^{p/\alpha}, \quad \lambda > 0,\ p < \alpha.$$

Recall for α-stable subordinator, L. Xu (2018):

$$\mathbb{E}\left(\int_0^T t^{-\theta}\,\mathrm{d}S_t\right)^p \le C_{\alpha,\theta,p}\,\boldsymbol{T}^{p(1/\alpha-\theta)}, \quad \theta < 1/\alpha,\ p \in (0,\alpha),$$

$$\mathbb{E}\left(\int_0^T \mathrm{e}^{-\lambda t}\,\mathrm{d}S_t\right)^p \le C_{\alpha,\lambda,p}, \quad \lambda > 0,\ p \in (0,\alpha).$$

Question:

Q3: **Optimality** for the moment estimates?

$$\mathbb{E}\left(\int_0^T t^{-\theta}\,\mathrm{d}S_t\right)^p = \frac{\Gamma\left(1-\frac{p}{\alpha}\right)}{(1-\alpha\theta)^{p/\alpha}\Gamma(1-p)}\,\boldsymbol{T}^{p(1/\alpha-\theta)}, \quad \theta < 1/\alpha,\ p < \alpha,$$

$$\mathbb{E}\left(\int_0^T \mathrm{e}^{-\lambda t}\,\mathrm{d}S_t\right)^p = \frac{\Gamma\left(1-\frac{p}{\alpha}\right)}{\Gamma(1-p)}\left(\frac{1-\mathrm{e}^{-\alpha\lambda T}}{\alpha\lambda}\right)^{p/\alpha}, \quad \lambda > 0,\ p < \alpha.$$

Our question (cont.)

Recall for α-stable subordinator, L. Xu (2018):

$$\mathbb{E}\left(\int_0^T t^{-\theta}\,\mathrm{d}S_t\right)^p \le C_{\alpha,\theta,p}\,\boldsymbol{T}^{p(1/\alpha-\theta)}, \quad \theta < 1/\alpha,\ p \in (0,\alpha),$$

$$\mathbb{E}\left(\int_0^T \mathrm{e}^{-\lambda t}\,\mathrm{d}S_t\right)^p \le C_{\alpha,\lambda,p}, \quad \lambda > 0,\ p \in (0,\alpha).$$

Question:

Q3: Optimality for the moment estimates?

$$\mathbb{E}\left(\int_0^T t^{-\theta}\,\mathrm{d}S_t\right)^p = \frac{\Gamma\left(1-\frac{p}{\alpha}\right)}{(1-\alpha\theta)^{p/\alpha}\Gamma(1-p)}\boldsymbol{T}^{p(1/\alpha-\theta)}, \quad \theta < 1/\alpha,\ p < \alpha,$$

$$\mathbb{E}\left(\int_0^T \mathrm{e}^{-\lambda t}\,\mathrm{d}S_t\right)^p = \frac{\Gamma\left(1-\frac{p}{\alpha}\right)}{\Gamma(1-p)}\left(\frac{1-\mathrm{e}^{-\alpha\lambda T}}{\alpha\lambda}\right)^{p/\alpha}, \quad \lambda > 0,\ p < \alpha.$$

Our question (cont.)

Recall for α-stable subordinator, L. Xu (2018):

$$\mathbb{E}\left(\int_0^T t^{-\theta}\,\mathrm{d}S_t\right)^p \le C_{\alpha,\theta,p}\,\boldsymbol{T}^{p(1/\alpha-\theta)}, \quad \theta < 1/\alpha,\ p \in (0,\alpha),$$

$$\mathbb{E}\left(\int_0^T \mathrm{e}^{-\lambda t}\,\mathrm{d}S_t\right)^p \le C_{\alpha,\lambda,p}, \quad \lambda > 0,\ p \in (0,\alpha).$$

Question:

Q3: Optimality for the moment estimates?

$$\mathbb{E}\left(\int_0^T t^{-\theta}\,\mathrm{d}S_t\right)^p = \frac{\Gamma\left(1-\frac{p}{\alpha}\right)}{(1-\alpha\theta)^{p/\alpha}\Gamma(1-p)}\boldsymbol{T}^{p(1/\alpha-\theta)}, \ \theta < 1/\alpha,\ p < \alpha,$$

$$\mathbb{E}\left(\int_0^T \mathrm{e}^{-\lambda t}\,\mathrm{d}S_t\right)^p = \frac{\Gamma\left(1-\frac{p}{\alpha}\right)}{\Gamma(1-p)}\left(\frac{1-\mathrm{e}^{-\alpha\lambda T}}{\alpha\lambda}\right)^{p/\alpha}, \quad \lambda > 0,\ p < \alpha.$$

Our question (cont.)

Recall for α-stable subordinator, L. Xu (2018):

$$\mathbb{E}\left(\int_0^T t^{-\theta}\,\mathrm{d}S_t\right)^p \le C_{\alpha,\theta,p}\,\boldsymbol{T}^{p(1/\alpha-\theta)}, \quad \theta < 1/\alpha,\ p \in (0,\alpha),$$

$$\mathbb{E}\left(\int_0^T \mathrm{e}^{-\lambda t}\,\mathrm{d}S_t\right)^p \le C_{\alpha,\lambda,p}, \quad \lambda > 0,\ p \in (0,\alpha).$$

Question:

Q3: Optimality for the moment estimates?

$$\mathbb{E}\left(\int_0^T t^{-\theta}\,\mathrm{d}S_t\right)^p = \frac{\Gamma\left(1-\frac{p}{\alpha}\right)}{(1-\alpha\theta)^{p/\alpha}\Gamma(1-p)}\,\boldsymbol{T}^{p(1/\alpha-\theta)}, \ \theta < 1/\alpha,\ p < \alpha,$$

$$\mathbb{E}\left(\int_0^T \mathrm{e}^{-\lambda t}\,\mathrm{d}S_t\right)^p = \frac{\Gamma\left(1-\frac{p}{\alpha}\right)}{\Gamma(1-p)}\left(\frac{1-\mathrm{e}^{-\alpha\lambda T}}{\alpha\lambda}\right)^{p/\alpha}, \quad \lambda > 0,\ p < \alpha.$$

Recall for α-stable subordinator, L. Xu (2018):

$$\mathbb{E}\left(\int_0^T t^{-\theta}\,\mathrm{d}S_t\right)^p \le C_{\alpha,\theta,p}\,\boldsymbol{T}^{p(1/\alpha-\theta)}, \quad \theta < 1/\alpha,\ p \in (0,\alpha),$$

$$\mathbb{E}\left(\int_0^T \mathrm{e}^{-\lambda t}\,\mathrm{d}S_t\right)^p \le C_{\alpha,\lambda,p}, \quad \lambda > 0,\ p \in (0,\alpha).$$

Question:

Q3: **Optimality** for the moment estimates?

$$\mathbb{E}\left(\int_0^T t^{-\theta}\,\mathrm{d}S_t\right)^p = \frac{\Gamma\left(1-\frac{p}{\alpha}\right)}{(1-\alpha\theta)^{p/\alpha}\Gamma(1-p)}\,\boldsymbol{T}^{p(1/\alpha-\theta)}, \ \theta < 1/\alpha,\ p < \alpha,$$

$$\mathbb{E}\left(\int_0^T \mathrm{e}^{-\lambda t}\,\mathrm{d}S_t\right)^p = \frac{\Gamma\left(1-\frac{p}{\alpha}\right)}{\Gamma(1-p)}\left(\frac{1-\mathrm{e}^{-\alpha\lambda T}}{\alpha\lambda}\right)^{p/\alpha}, \quad \lambda > 0,\ p < \alpha.$$

Subordinator

- A subordinator S_t is a non-decreasing Lévy process on $\mathbb{R}_+$.
- A subordinator S_t is uniquely determined by

$$\mathbb{E}\,\mathrm{e}^{-uS_t} = \mathrm{e}^{-t\phi(u)}, \quad u > 0,\, t \geq 0.$$

- Lévy-Khintchine representation of Bernstein function:

$$\phi(u) = a + bu + \int_{(0,\infty)} \left(1 - \mathrm{e}^{-ux}\right) \nu(\mathrm{d}x),$$

where $a, b \geq 0$, $\int_{(0,\infty)} (x \wedge 1)\, \nu(\mathrm{d}x) < \infty$.

- subordinator S_t $\leftrightarrow$ Bernstein function ϕ $\leftrightarrow$ (a, b, ν)

one-to-one

Subordinator

- A subordinator S_t is a non-decreasing Lévy process on $\mathbb{R}_+$.
- A subordinator S_t is uniquely determined by

$$\mathbb{E}\,\mathrm{e}^{-uS_t} = \mathrm{e}^{-t\phi(u)}, \quad u > 0,\, t \geq 0.$$

- Lévy-Khintchine representation of Bernstein function:

$$\phi(u) = a + bu + \int_{(0,\infty)} \left(1 - \mathrm{e}^{-ux}\right) \nu(\mathrm{d}x),$$

where $a, b \geq 0$, $\int_{(0,\infty)} (x \wedge 1)\, \nu(\mathrm{d}x) < \infty$.

- subordinator S_t $\leftrightarrow$ Bernstein function ϕ $\leftrightarrow$ (a, b, ν)

one-to-one

Subordinator

- A subordinator S_t is a non-decreasing Lévy process on $\mathbb{R}_+$.
- A subordinator S_t is uniquely determined by

$$\mathbb{E}\,\mathrm{e}^{-uS_t} = \mathrm{e}^{-t\phi(u)}, \quad u > 0,\, t \geq 0.$$

- Lévy-Khintchine representation of Bernstein function:

$$\phi(u) = a + bu + \int_{(0,\infty)} \left(1 - \mathrm{e}^{-ux}\right) \nu(\mathrm{d}x),$$

where $a, b \geq 0$, $\int_{(0,\infty)} (x \wedge 1)\, \nu(\mathrm{d}x) < \infty$.

- subordinator S_t $\leftrightarrow$ Bernstein function ϕ $\leftrightarrow$ (a, b, ν)

one-to-one

Subordinator

- A subordinator S_t is a non-decreasing Lévy process on $\mathbb{R}_+$.
- A subordinator S_t is uniquely determined by

$$\mathbb{E}\,\mathrm{e}^{-uS_t} = \mathrm{e}^{-t\phi(u)}, \quad u > 0,\ t \geq 0.$$

- Lévy-Khintchine representation of Bernstein function:

$$\phi(u) = a + bu + \int_{(0,\infty)} \left(1 - \mathrm{e}^{-ux}\right) \nu(\mathrm{d}x),$$

where $a, b \geq 0$, $\int_{(0,\infty)} (x \wedge 1)\, \nu(\mathrm{d}x) < \infty$.

- subordinator S_t $\leftrightarrow$ Bernstein function ϕ $\leftrightarrow$ (a, b, ν)

one-to-one

- α-stable subordinator
  Bernstein function: $\phi(u) = u^\alpha$
  Lévy triplet: $a = b = 0, \nu(\mathrm{d}x) = \frac{\alpha}{\Gamma(1-\alpha)} x^{-1-\alpha}\,\mathrm{d}x$
- Relativistic stable subordinator
  Bernstein function: $\phi(u) = (u+1)^\alpha - 1$
  Lévy triplet: $a = b = 0, \nu(\mathrm{d}x) = \frac{\alpha}{\Gamma(1-\alpha)} \mathrm{e}^{-x} x^{-1-\alpha}\,\mathrm{d}x$
- Gamma subordinator
  Bernstein function: $\phi(u) = \log(1+u)$
  Lévy triplet: $a = b = 0, \nu(\mathrm{d}x) = \mathrm{e}^{-x} x^{-1}\,\mathrm{d}x$
- Geometric stable subordinator, Bessel subordinator, $\cdots$

Examples of subordinators

- α-stable subordinator
  Bernstein function: $\phi(u) = u^\alpha$
  Lévy triplet: $a = b = 0, \nu(\mathrm{d}x) = \frac{\alpha}{\Gamma(1-\alpha)} x^{-1-\alpha}\,\mathrm{d}x$

- Relativistic stable subordinator
  Bernstein function: $\phi(u) = (u+1)^\alpha - 1$
  Lévy triplet: $a = b = 0, \nu(\mathrm{d}x) = \frac{\alpha}{\Gamma(1-\alpha)} \mathrm{e}^{-x} x^{-1-\alpha}\,\mathrm{d}x$

- Gamma subordinator
  Bernstein function: $\phi(u) = \log(1+u)$
  Lévy triplet: $a = b = 0, \nu(\mathrm{d}x) = \mathrm{e}^{-x} x^{-1}\,\mathrm{d}x$

- Geometric stable subordinator, Bessel subordinator, $\cdots$

Examples of subordinators

- α-stable subordinator
  Bernstein function: $\phi(u) = u^\alpha$
  Lévy triplet: $a = b = 0, \nu(\mathrm{d}x) = \frac{\alpha}{\Gamma(1-\alpha)} x^{-1-\alpha}\,\mathrm{d}x$

- Relativistic stable subordinator
  Bernstein function: $\phi(u) = (u+1)^\alpha - 1$
  Lévy triplet: $a = b = 0, \nu(\mathrm{d}x) = \frac{\alpha}{\Gamma(1-\alpha)} \mathrm{e}^{-x} x^{-1-\alpha}\,\mathrm{d}x$

- Gamma subordinator
  Bernstein function: $\phi(u) = \log(1+u)$
  Lévy triplet: $a = b = 0, \nu(\mathrm{d}x) = \mathrm{e}^{-x} x^{-1}\,\mathrm{d}x$

- Geometric stable subordinator, Bessel subordinator, $\cdots$

Examples of subordinators

- α-stable subordinator
  Bernstein function: $\phi(u) = u^\alpha$
  Lévy triplet: $a = b = 0, \nu(\mathrm{d}x) = \frac{\alpha}{\Gamma(1-\alpha)} x^{-1-\alpha}\,\mathrm{d}x$

- Relativistic stable subordinator
  Bernstein function: $\phi(u) = (u+1)^\alpha - 1$
  Lévy triplet: $a = b = 0, \nu(\mathrm{d}x) = \frac{\alpha}{\Gamma(1-\alpha)} \mathrm{e}^{-x} x^{-1-\alpha}\,\mathrm{d}x$

- Gamma subordinator
  Bernstein function: $\phi(u) = \log(1+u)$
  Lévy triplet: $a = b = 0, \nu(\mathrm{d}x) = \mathrm{e}^{-x} x^{-1}\,\mathrm{d}x$

- Geometric stable subordinator, Bessel subordinator, $\cdots$

Our result: a zero-one law

Theorem (D.-Schilling-Xu, arXiv: 2009.04785)

Let S_t be a subordinator with Laplace exponent ϕ, and $f : [0,\infty) \to [0,\infty)$ measurable. Then the following are equivalent:

i) $$\mathbb{P}\left(\int_0^\infty f(t)\,\mathrm{d}S_t < \infty\right) > 0.$$

ii) $$\mathbb{P}\left(\int_0^\infty f(t)\,\mathrm{d}S_t < \infty\right) = 1.$$

iii) $$\int_0^\infty \phi\big(f(t)\big)\,\mathrm{d}t < \infty.$$

Remark: For bounded f, Kolmogorov's zero-one law.

Our result: a zero-one law

Theorem (D.-Schilling-Xu, arXiv: 2009.04785)

Let S_t be a subordinator with Laplace exponent ϕ, and $f : [0,\infty) \to [0,\infty)$ measurable. Then the following are equivalent:

i) $$\mathbb{P}\left(\int_0^\infty f(t)\,\mathrm{d}S_t < \infty\right) > 0.$$

ii) $$\mathbb{P}\left(\int_0^\infty f(t)\,\mathrm{d}S_t < \infty\right) = 1.$$

iii) $$\int_0^\infty \phi\big(f(t)\big)\,\mathrm{d}t < \infty.$$

Remark: For bounded f, Kolmogorov's zero-one law.

Related results

Proposition(J. Bertoin, 1999)

Let $f : [0, \infty) \to [0, \infty)$ be non-increasing. Then the following are equivalent:

i) $\mathbb{P}\left(\int_0^\infty f(S_t)\,\mathrm{d}t < \infty\right) > 0.$

ii) $\mathbb{P}\left(\int_0^\infty f(S_t)\,\mathrm{d}t < \infty\right) = 1.$

iii) $\int_0^\infty \mathbb{E} f(S_t)\,\mathrm{d}t < \infty.$

Remark: P.-S. Li & X. Zhou 2018,
a related zero-one law for spectrally positive Lévy process.

Proposition(J. Bertoin, 1999)

Let $f : [0, \infty) \to [0, \infty)$ be non-increasing. Then the following are equivalent:

i) $\mathbb{P}\left(\int_0^\infty f(S_t)\,\mathrm{d}t < \infty\right) > 0.$

ii) $\mathbb{P}\left(\int_0^\infty f(S_t)\,\mathrm{d}t < \infty\right) = 1.$

iii) $\int_0^\infty \mathbb{E} f(S_t)\,\mathrm{d}t < \infty.$

Remark: P.-S. Li & X. Zhou 2018,
a related zero-one law for spectrally positive Lévy process.

Proposition

Let $f : [0,\infty) \to [0,\infty)$ be measurable (non-trivial). Then

$$\mathbb{E}\left(\int_0^\infty f(t)\,\mathrm{d}S_t\right)^p = \begin{cases} \dfrac{\Gamma\left(1-\frac{p}{\alpha}\right)}{\Gamma(1-p)}\left(\int_0^\infty f(t)^\alpha\,\mathrm{d}t\right)^{p/\alpha}, & \text{if } p<\alpha, \\ \infty, & \text{if } p\geq\alpha. \end{cases}$$

Moment estimates for general case

Proposition

If

$$\liminf_{s\to\infty} \frac{\phi(2s)}{\phi(s)} > 1,$$

then for all $p < 0$ and $\theta \geq 0$,

$$\mathbb{E}\left(\int_0^T t^{-\theta}\, \mathrm{d}S_t\right)^p \leq CT^{-p\theta}\left[\phi^{-1}\left(\frac{1}{T}\right)\right]^{-p}, \quad T \in (0,1].$$

Typical examples:

$$\phi(s) = s^\alpha \log^\beta(1+s), \qquad 0 < \alpha < 1,\ 0 \leq \beta \leq 1-\alpha$$

$$\phi(s) = s^\alpha \log^{-\beta}(1+s), \qquad 0 \leq \beta \leq \alpha < 1$$

$$\phi(s) = s(1+s)^{-\alpha}, \qquad 0 < \alpha < 1$$

Moment estimates for general case

Proposition

If

$$\liminf_{s\to\infty} \frac{\phi(2s)}{\phi(s)} > 1,$$

then for all $p < 0$ and $\theta \geq 0$,

$$\mathbb{E}\left(\int_0^T t^{-\theta}\, \mathrm{d}S_t\right)^p \leq C T^{-p\theta} \left[\phi^{-1}\left(\frac{1}{T}\right)\right]^{-p}, \quad T \in (0,1].$$

Typical examples:

$$\phi(s) = s^\alpha \log^\beta(1+s), \qquad 0<\alpha<1,\ 0\leq\beta\leq 1-\alpha$$

$$\phi(s) = s^\alpha \log^{-\beta}(1+s), \qquad 0\leq\beta\leq\alpha<1$$

$$\phi(s) = s(1+s)^{-\alpha}, \qquad 0<\alpha<1$$

Moment estimates for general case

Proposition

If

$$\liminf_{s\to\infty} \frac{\phi(2s)}{\phi(s)} > 1,$$

then for all $p < 0$ and $\theta \geq 0$,

$$\mathbb{E}\left(\int_0^T t^{-\theta}\,\mathrm{d}S_t\right)^p \leq CT^{-p\theta}\left[\phi^{-1}\left(\frac{1}{T}\right)\right]^{-p}, \quad T \in (0,1].$$

Remark: moment estimates for

Lévy process: D.-Schilling, SPA 2015, EJP 2017

Lévy-type process: Kühn, SPA 2017

Discrete subordinator: D., JTP 2020

Moment estimates for general case

Proposition

If

$$\liminf_{s\to\infty} \frac{\phi(s)}{\log s} > 0, \quad \text{and} \quad \liminf_{s\downarrow 0} \frac{\phi(2s)}{\phi(s)} > 1,$$

then for all $p < 0$ and $\theta \geq 0$,

$$\mathbb{E}\left(\int_0^T t^{-\theta}\, \mathrm{d}S_t\right)^p \leq C T^{-p\theta} \left[\phi^{-1}\left(\frac{1}{T}\right)\right]^{-p}, \quad T \geq 1.$$

Typical examples:

$$\phi(s) = s^\alpha \log^\beta(1+s), \qquad 0 < \alpha < 1,\ 0 \leq \beta \leq 1-\alpha$$

$$\phi(s) = s^\alpha \log^{-\beta}(1+s), \qquad 0 \leq \beta \leq \alpha < 1$$

$$\phi(s) = s(1+s)^{-\alpha}, \qquad 0 < \alpha < 1$$

Moment estimates for general case

Proposition

If

$$\liminf_{s\to\infty} \frac{\phi(s)}{\log s} > 0, \quad \text{and} \quad \liminf_{s\downarrow 0} \frac{\phi(2s)}{\phi(s)} > 1,$$

then for all $p < 0$ and $\theta \geq 0$,

$$\mathbb{E}\left(\int_0^T t^{-\theta}\, \mathrm{d}S_t\right)^p \leq CT^{-p\theta}\left[\phi^{-1}\left(\frac{1}{T}\right)\right]^{-p}, \quad T \geq 1.$$

Typical examples:

$$\phi(s) = s^\alpha \log^\beta(1+s), \qquad 0 < \alpha < 1,\ 0 \leq \beta \leq 1-\alpha$$

$$\phi(s) = s^\alpha \log^{-\beta}(1+s), \qquad 0 \leq \beta \leq \alpha < 1$$

$$\phi(s) = s(1+s)^{-\alpha}, \qquad 0 < \alpha < 1$$

- Consider the SPDE

$$\mathrm{d}X_t = -AX_t\mathrm{d}t + F(X_t)\mathrm{d}t + Q(X_{t-})\mathrm{d}W_{S_t}.$$

- Solution

$$X_t = \mathrm{e}^{-At}X_0 + \int_0^t \mathrm{e}^{-(t-s)A}F(X_s)\,\mathrm{d}s + \underbrace{\int_0^t \mathrm{e}^{-(t-s)A}Q(X_{s-})\,\mathrm{d}W_{S_s}}_{=:Z_t}$$

- Invariant measure, accessibility, Galerkin approximation

- Consider the SPDE

$$\mathrm{d}X_t = -AX_t\mathrm{d}t + F(X_t)\mathrm{d}t + Q(X_{t-})\mathrm{d}W_{S_t}.$$

- Solution

$$X_t = \mathrm{e}^{-At}X_0 + \int_0^t \mathrm{e}^{-(t-s)A}F(X_s)\,\mathrm{d}s + \underbrace{\int_0^t \mathrm{e}^{-(t-s)A}Q(X_{s-})\,\mathrm{d}W_{S_s}}_{=:Z_t}$$

- Invariant measure, accessibility, Galerkin approximation

- Consider the SPDE

$$\mathrm{d}X_t = -AX_t\mathrm{d}t + F(X_t)\mathrm{d}t + Q(X_{t-})\mathrm{d}W_{S_t}.$$

- Solution

$$X_t = \mathrm{e}^{-At}X_0 + \int_0^t \mathrm{e}^{-(t-s)A}F(X_s)\,\mathrm{d}s + \underbrace{\int_0^t \mathrm{e}^{-(t-s)A}Q(X_{s-})\,\mathrm{d}W_{S_s}}_{=:Z_t}$$

- Invariant measure, accessibility, Galerkin approximation

S(P)DEs driven by subordinate Brownian motion

Malliavin calculus (Kusuoka, 2009)

Bismut-Elworthy-Li's formula (X. Zhang, SPA 2013)

integration by parts formula (F.-Y. Wang, 2014)

Wang's Harnack inequality (J. Wang & F.-Y. Wang, JMAA 2014)

asymptotics (R. Wang & L. Xu, SPA 2018)

......

A maximal inequality for Z_t

$$Z_t = \int_0^t \mathrm{e}^{-(t-s)A} Q(X_{s-})\, \mathrm{d}W_{S_s}.$$

Proposition

If

$$0 < p < 2\log_2\left(\liminf_{s\to 0} \frac{\phi(2s)}{\phi(s)}\right),$$

then there exists $C = C(p, \|Q\|_{\mathrm{HS},\infty}) > 0$ such that

$$\mathbb{E}\left[\sup_{0\le t\le T} |Z_t|^p\right] \le C\left[\phi^{-1}\left(\frac{1}{T}\right)\right]^{-\frac{p}{2}} \quad \text{for all } T \ge 1.$$

A small ball probability for Z_t

$$Z_t = \int_0^t \mathrm{e}^{-(t-s)A} Q(X_{s-})\, \mathrm{d}W_{S_s}.$$

Proposition

For all $\delta > 0$,

$$\mathbb{P}\left(\sup_{0\le t\le T} |Z_t| < \delta\right) > 0.$$

$$\mathrm{d}X_t = -AX_t\mathrm{d}t + F(X_t)\mathrm{d}t + Q(X_{t-})\mathrm{d}W_{S_t}.$$

Theorem

If

$$\liminf_{s\to 0} \frac{\phi(2s)}{\phi(s)} > 1,$$

then this system admits an invariant measure.

Accessibility

$$\mathrm{d}X_t = -AX_t\mathrm{d}t + F(X_t)\mathrm{d}t + Q(X_{t-})\mathrm{d}W_{S_t}.$$

Theorem (Accessible to zero)

If there exists some $\delta > 0$ such that

$$\int_{0+} \phi(s^{-2\delta})\,\mathrm{d}s < \infty, \quad \sup_{x\in H} \left\| Q(x)^{-1}\mathrm{e}^{-tA} \right\| \le Ct^{-\delta},$$

then for all $x \in H$, $\epsilon > 0$, $T > 0$,

$$\mathbb{P}\left(|X_T(x)| < \epsilon\right) > 0.$$

Remark: Priola-Shirikyan-Xu-Zabczyk, SPA 2012

Accessibility

$$\mathrm{d}X_t = -AX_t\mathrm{d}t + F(X_t)\mathrm{d}t + Q(X_{t-})\mathrm{d}W_{S_t}.$$

Theorem (Accessible to zero)

If there exists some $\delta > 0$ such that

$$\int_{0+} \phi(s^{-2\delta})\,\mathrm{d}s < \infty, \quad \sup_{x\in H} \left\|Q(x)^{-1}\mathrm{e}^{-tA}\right\| \le Ct^{-\delta},$$

then for all $x \in H$, $\epsilon > 0$, $T > 0$,

$$\mathbb{P}\left(|X_T(x)| < \epsilon\right) > 0.$$

Remark: Priola-Shirikyan-Xu-Zabczyk, SPA 2012

Thanks for Your Attention!